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OPTIMIZATION OF STRUCTURE AND MATERIAL PROPERTIES FOR SOLIDS COMPOSED OF SOFTENING MATERIAL

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Abstract—Recent results on the design of material properties in the context of global structural optimization provide, in analytical form, a prediction of the optimal *material tensor* distributions for two or three dimensional continuum structures. The model developed for that purpose is extended here to cover the design of a structure and associated material properties for a system composed of a generic form of *nonlinear softening material*. As was established in the earlier study on design with linear materials, the formulation for combined "material and structure" design with softening materials can be expressed as a convex problem. However, in contrast to the case with linear material, the optimal distribution of material properties predicted in the nonlinear problem depends on the magnitude of load. Computational solutions are presented for several example problems, showing how the optimal designs vary with different values assigned to data that fix the load and material parameters.

1. INTRODUCTION

The purpose here is to treat, in analytical form, the design problem for simultaneous prediction of material properties and structural layout. In the present approach, this is accomplished simply by considering the design to be characterized in the formulation via a free parametrization of the rigidity tensor of the material. Formulations of this kind have been demonstrated recently for structures composed of linearly elastic material, in both a single purpose and multiple purpose design context [Bendsøe *et al.* (1993), (1994)]. In these studies the rigidity tensor is allowed to range over all positive, semi-definite tensors, and the design resource (or total cost) is measured through invariants of the tensor. The objective was taken to be "design for minimum compliance". Within this formulation a *material optimization problem* can be identified, and thus the optimal local form of the material tensor can be derived. Once the optimal local material properties are determined, the original design problem can be expressed as a simpler equivalent design problem statement involving only the global distribution of resource. In this way the problem size is reduced considerably. For a single loading condition this auxiliary problem takes on a simple form, one similar to that of a variable thickness sheet design problem.

In the developments to follow we will describe an extension of this free material design formulation to the design of a structure composed of a generic form of *nonlinear softening material*. The relevant mechanics are represented in the new formulation in terms of a generalized complementary energy principle developed recently for modelling the equilibrium analysis of such structures [Taylor (1993a)]. For present purposes the design objective is likewise based on complementary energy. Net material properties of the softening medium reflect a superposition of properties associated with each of a number of material constituents, and the collection of these properties, expressed through the rigidity tensors for each of these constituents, provides the problem with a set of design parameters. Analytical forms for the optimal material tensor and the global distribution of material can be derived, in much the same way as was indicated above for design with linear materials, and thus the design parameters can be removed from the problem. The design parameters are then hidden parameters and will be given entirely in terms of the stress fields of an auxiliary, reduced problem which is an *equilibrium only* problem (albeit one with a non-linear, nonsmooth (optimal) complementary energy functional). Alternatively, by solving analytically only for the optimal *local* properties, the resulting reduced problem is a smooth and convex problem combining equilibrium analysis and the determination of the optimal distribution of bulk resource. This problem is tractable, and computational examples are presented to show the form of results predicted for optimal material distribution.

The work presented in this paper represents a natural extension of the recent developments on simultaneous design of material and structure [see e.g. Bendsøe *et al.* (1994 a,b)]. It constitutes, as well, a natural progression of developments in modelling for optimal design with advanced materials, and from treatments of topology design using homogenization modelling (see, e.g., the collection of papers in Bendsøe and Mota Soares (1993), Pedersen (1993)). For models that employ the homogenization modelling for design parametrization, the optimal local material parameters can be related directly to a suitable microstructure, as demonstrated in Allaire and Kohn (1993), Jog *et al.* (1994). In the context of the present free design parametrization, a different form of "local structure" is required for the realization of material tensors. Examples of microstructures suitable for this purpose are described in Milton and Cherkaev (1993) and in Sigmund (1994). These forms of local structure are not unique, nor are they necessarily of significance here other than to establish the quality of "realizability".

2. PROBLEM STATEMENT

As indicated in the introduction, availability of an extremum problem formulation for the analysis part of the problem is what makes it possible to treat the design of nonlinear materials conveniently. The type of formulation used in the following development, which amounts to a generalized form of complementary energy principle, is presented in detail in Taylor (1993a). It is described briefly here to set the stage for the subsequent extension to cover design. The portrayal of a general form of nonlinear softening material relies on a feature in the model that has total stress expressed via a superposition of an arbitrary number of independent (constituent) fields. Each such constituent field is represented to be arbitrarily heterogeneous and anisotropic, and constituent stresses may be constrained to lie within a limiting surface. Overall material properties are determined through the model, once the parameters governing each of the constituent fields are specified as data.

The formulation for equilibrium analysis is stated here first in terms of mixed stress and deformation fields. With the superposition of P softening components and one strictly linear basis component to make up the total stress, the problem has the form:

$$
\begin{aligned}
&\max_{\alpha,\sigma^P,u} \alpha \\
&\text{subject to:} \\
&\operatorname{div}\left(F_{ijkl}\varepsilon_{kl}(u)+\sum_{p=1}^{P}\sigma_{ij}^{p}\right)+\alpha f=0 \\
&\left(F_{ijkl}\varepsilon_{kl}(u)+\sum_{p=1}^{P}\sigma_{ij}^{p}\right)\cdot n=\alpha t \quad \text{on} \quad \Gamma_T, u=0 \quad \text{on} \quad \Gamma_0 \\
&\sigma^p\in K_p, p=1,\ldots,P \\
&\frac{1}{2}\int_{\Omega}\left(F_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)+\sum_{p=1}^{P}C_{ijkl}^{p}\sigma_{ij}^{p}\sigma_{kl}^{p}\right)\mathrm{d}\Omega\leqslant\Phi.
\end{aligned}
\tag{A}
$$

Here $C^p_{ijkl} = E^{p^{-1}}_{ijkl}$ are the compliance tensors for the P softening components and F_{ijkl} is the rigidity tensor for the strictly linear component. Factor α provides for the description of loads in the form of proportional loading. The stresses for the softening components are denoted σ^p_{ij} and the displacement of the continuum by u; $\varepsilon_{ij}(u)$ symbolizes strains linear in displacement u. The structure is subject to body force f and surface traction t. Boundary $\partial\Omega$ of the structure is made up of the traction boundary Γ_T and the prescribed displacements boundary Γ_0, such that $\partial\Omega = \Gamma = \Gamma_T \cup \Gamma_0$, $\Gamma_T \cap \Gamma_0 = \varnothing$. [For simplicity, problem statement (A) is written here for the case where the displacement on Γ_0 has value zero]. Finally, the convex sets of admissible stresses σ^p_{ij} for the softening components are denoted by K_p. Problem (A) is written for a *given material*, and for the analysis problem which it models the combined compliance tensors, rigidity tensor, and the information that serves to define sets K_p altogether comprise the data which govern overall material properties. For the "design of material properties" problem to be considered below, one or more of these material property tensors are treated as design variables.

As an alternative, the basic equilibrium analysis problem (A) can be stated in terms of stresses alone as:

$$\max_{\alpha,\sigma^p,\gamma} \alpha$$

subject to:

$$\operatorname{div}\left(\gamma_{ij} + \sum_{p=1}^{P} \sigma^p_{ij}\right) + \alpha f = 0$$

$$\left(\gamma_{ij} + \sum_{p=1}^{P} \sigma^p_{ij}\right) \cdot n = \alpha t \quad \text{on} \quad \Gamma_T,$$

$$\sigma^p \in K_p, p = 1, \ldots, P$$

$$\frac{1}{2}\int_\Omega \left(F^{-1}_{ijkl}\gamma_{ij}\gamma_{kl} + \sum_{p=1}^{P} C^p_{ijkl}\sigma^p_{ij}\sigma^p_{kl}\right) \mathrm{d}\Omega \leqslant \Phi. \tag{A'}$$

This form of the problem statement is a parametrized complementary energy formulation for the general softening material. The solution to (A′) predicts a bound to the equilibrium load within the limit Φ on total complementary energy.

Following the recent studies on design of optimal material parameters cited in the introduction, it is natural to consider here the extension covering design of the nonlinear material for maximization of load carrying capacity. Using the *rigidity tensors* as free design variables, this design problem has the form:

$$\sup_{E^p,F}\max_{\alpha,\sigma^p,\gamma} \alpha$$

subject to:

$$\operatorname{div}\left(\gamma_{ij} + \sum_{p=1}^{P} \sigma^p_{ij}\right) + \alpha f = 0$$

$$\left(\gamma_{ij} + \sum_{p=1}^{P} \sigma^p_{ij}\right) \cdot n = \alpha t \quad \text{on} \quad \Gamma_T,$$

$$\sigma^p \in K_p, p = 1, \ldots, P$$

$$\frac{1}{2}\int_\Omega \left(F^{-1}_{ijkl}\gamma_{ij}\gamma_{kl} + \sum_{p=1}^{P} E^{p^{-1}}_{ijkl}\sigma^p_{ij}\sigma^p_{kl}\right) \mathrm{d}\Omega \leqslant \Phi$$

$$E^p > 0, F > 0,$$

$$\int_\Omega \Psi(F)\,\mathrm{d}\Omega \leqslant V_0\,; \int_\Omega \Psi(E^p)\,\mathrm{d}\Omega \leqslant V_p, p = 1, \ldots P. \tag{P}$$

Here the design is to be optimal with respect to all positive definite rigidity tensors representing any (anisotropic) material, and "material resource" is measured in terms of invariants (symbolized by Ψ in problem statement (P)) of these tensors. In the statement (P) we take the supremum over the rigidity tensors, as we are using a stress based formulation; this is inherent to the analysis case under study. For pure displacement based formulations [see e.g., Bendsøe *et al.* (1993)], the design optimization can be performed over all positive *semi*-definite rigidity tensors. The difference in approach is not reflected in the solution, but rather relates to the matter of having the problem expressed in a well posed form.

We choose here to use either the trace or the Frobenius norm to measure resource for all tensors in the formulation, and this means that the invariants $\Psi(F)$; $\Psi(E^p)$ in (P), hereafter represented as "resource densities ρ", are given as:

$$\rho_{tr}(F) = F_{ijij}; \quad \rho_{tr}(E^p) = E^p_{ijij}, \quad p = 1, \ldots, P$$

for the trace measure and

$$\rho_F(F) = \sqrt{F_{ijkl}F_{ijkl}}; \quad \rho_F(E^p) = \sqrt{E^p_{ijkl}E^p_{ijkl}}, \quad p = 1, \ldots, P$$

for the Frobenius norm. Note that these measures are homogeneous of degree one. Thus, compared to the conventional 2D problem for the design of material distribution in a sheet (where total cost is proportional to the volume of material), the above "cost measures" correspond in their role to the sheet thickness.

In the formulation above it is assumed that the sets K_p of admissible softening components σ^p of the total stress are *design independent*. Thus, the solution to problem (P) predicts the optimal distribution of rigidities within these specified softening limits (optimal design with the limits themselves as design variables is treated for arbitrary trussed structures in Taylor (1993b)).

In the case of truss structures modelled as above, design for maximum load carrying capacity using member cross-sectional areas as design variables has been studied by Bendsøe *et al.* (1993) for the case of an elasto-plastic formulation. Truss design for the general softening material is reported in Taylor and Logo (1993) and Taylor (1993b).

Problem (P) is simultaneously convex in stress and design variables (the function x^2/y, $y > 0$ is convex in (x, y)). This implies that problem (P) (up to a rescaling factor on the loading) is equivalent to the convex problem:

$$\inf_{E^p, F} \min_{\sigma^p, \gamma} \frac{1}{2}\int_\Omega \left(F^{-1}_{ijkl}\gamma_{ij}\gamma_{kl} + \sum_{p=1}^{P} E^{p^{-1}}_{ijkl}\sigma^p_{ij}\sigma^p_{kl} \right) \mathrm{d}\Omega$$

subject to:

$$\operatorname{div}\left(\gamma_{ij} + \sum_{p=1}^{P} \sigma^p_{ij}\right) + \bar{\alpha} f = 0$$

$$\left(\gamma_{ij} + \sum_{p=1}^{P} \sigma^p_{ij}\right) \cdot n = \bar{\alpha} t \quad \text{on} \quad \Gamma_T,$$

$$\sigma^p \in K_p, p = 1, \ldots, P$$

$$E^p > 0, F > 0,$$

$$\int_\Omega \Psi(F)\,\mathrm{d}\Omega \leqslant V_0; \int_\Omega \Psi(E^p)\,\mathrm{d}\Omega \leqslant V_p, p = 1, \ldots P. \tag{P2}$$

Here $\bar{\alpha}$ stands for a specified value of the load factor.

3. ANALYTICAL REDUCTION OF THE PROBLEM

For the case where no softening constituents are present, problem (P2) is precisely a complementary energy based formulation of the minimum compliance design problem with free material design; this problem is described in detail in Bendsøe *et al.* (1993b). Along the lines of the modelling used in that study, parameters that describe the structure are now divided into two groups, namely those parameters that measure the amount of resource assigned to each point of the domain, and a second set that delineates how this resource is used to form the local material tensor. This provides for the following multi-level formulation of the problem:

$$\inf_{\substack{\rho_p,\rho_0 \\ \int_\Omega \rho_0 \mathrm{d}\Omega \leq V_0; \int_\Omega \rho_p \mathrm{d}\Omega \leq V_p}} \inf_{\substack{E^p>0, F>0, \\ \Psi(F)=\rho_0, \\ \Psi(E^p)=\rho_p}} \left\{ \begin{array}{l} \min_{\sigma^p,\gamma} \dfrac{1}{2}\displaystyle\int_\Omega \left(F_{ijkl}^{-1}\gamma_{ij}\gamma_{kl} + \sum_{p=1}^{P} E_{ijkl}^{p^{-1}} \sigma_{ij}^p \sigma_{kl}^p \right) \mathrm{d}\Omega \\ \text{subject to:} \\ \operatorname{div}\left(\gamma_{ij} + \displaystyle\sum_{p=1}^{P} \sigma_{ij}^p\right) + \bar{\alpha} f = 0 \\ \left(\gamma_{ij} + \displaystyle\sum_{p=1}^{P} \sigma_{ij}^p\right) \cdot n = \bar{\alpha} t \quad \text{on} \quad \Gamma_T, \\ \sigma^p \in K_p, p = 1, \ldots, P \end{array} \right\} \tag{P3}$$

Here the statical admissibility conditions of the inner problem are independent of the design variables. Thus minimization with respect to the pointwise variation of the rigidity tensors can be represented in the form:

$$\inf_{\substack{E^p>0, F>0, \\ \Psi(F)=\rho_0, \\ \Psi(E^p)=\rho_p}} \left\{ F_{ijkl}^{-1}\gamma_{ij}\gamma_{kl} + \sum_{p=1}^{P} E_{ijkl}^{p^{-1}} \sigma_{ij}^p \sigma_{kl}^p \right\}. \tag{P4}$$

This characterization is consistent with the assumption of pointwise independent variation of the tensors within fixed values ρ_0, ρ_p of resource. This in turn justifies minimization of the local measure in (P4) at each point of the structure. In problem (P4) the tensors F, E^p are independent, so we can take the infimum of each term independently. That problem has been studied in Bendsøe *et al.* (1993), represented there in a strain formulation, and from the results of that reference or by direct inspection (using a spectral decomposition of the rigidity tensor, see Appendix) we can conclude that we have

$$\inf_{E>0, \Psi(E)=\rho} E_{ijkl}^{-1} \sigma_{ij}\sigma_{kl} = \frac{1}{\rho} \sigma_{ij}\sigma_{ij} \tag{P5}$$

for any stress field and any rigidity tensor. This result applies for both the trace and Frobenius norm measures of resource. Note that the optimal energy expression in (P5) coincides with the energy of a linearly elastic, zero-Poisson-ratio material with a density of material equal to ρ. The infimum in (P5) is not achieved, but the infimum can be realized with the singular compliance tensor:

$$C_{ijkl} = \frac{1}{\rho} \frac{1}{\sigma_{pq}\sigma_{pq}} \sigma_{ij}\sigma_{kl}. \tag{MAT}$$

This corresponds to a singular *orthotropic* material, with axes of orthotropy aligned with the direction of principal stresses for the field σ_{ij}, and with only one non-zero eigenvalue. In terms of the rigidity tensors, a minimizing sequence can be constructed of non-singular tensors, e.g., see Appendix.

With the introduction of (P5) into (P4), the problem (P3) can now be reduced to the convex problem:

$$\inf_{\rho_p,\rho_0} \min_{\sigma^p,\gamma} \frac{1}{2}\int_\Omega \left(\frac{1}{\rho_0}\gamma_{ij}\gamma_{ij} + \sum_{p=1}^{P}\frac{1}{\rho_p}\sigma^p_{ij}\sigma^p_{ij}\right) d\Omega$$

subject to:

$$\operatorname{div}\left(\gamma_{ij} + \sum_{p=1}^{P}\sigma^p_{ij}\right) + \bar{\alpha}f = 0$$

$$\left(\gamma_{ij} + \sum_{p=1}^{P}\sigma^p_{ij}\right)\cdot n = \bar{\alpha}t \quad \text{on} \quad \Gamma_T,$$

$$\sigma^p \in K_p, p = 1,\ldots,P$$

$$\int_\Omega \Psi(F)\, d\Omega \leqslant V_0;\ \int_\Omega \Psi(E^p)\, d\Omega \leqslant V_p, p = 1,\ldots P. \tag{P6}$$

In (P6) the energy measure for each constituent corresponds to the complementary energy of a linear elastic, zero-Poisson-ratio material of density equal to the locally assigned resource value.

In problem (P6) we can solve for the resource densities, facilitated by the fact that the statical admissibility conditions of the inner equilibrium problem are independent of the design variables. From the necessary conditions of optimality (and the fact that the resource constraints are active), it is then straightforward to compute the resulting optimal resource densities as:

$$\rho_0 = V_0\sqrt{\gamma_{ij}\gamma_{ij}}\bigg/\int_\Omega \sqrt{\gamma_{ij}\gamma_{ij}}\, d\Omega;\quad \rho_p = V_p\sqrt{\sigma^p_{ij}\sigma^p_{ij}}\bigg/\int_\Omega \sqrt{\sigma^p_{ij}\sigma^p_{ij}}\, d\Omega. \tag{O1}$$

With the insertion of this result in problem statement (P6), the equivalent but now design independent problem takes the form:

$$\min_{\sigma^p,\gamma} \frac{1}{2V_0}\left[\int_\Omega \sqrt{\gamma_{ij}\gamma_{ij}}\, d\Omega\right]^2 + \sum_{p=1}^{P}\frac{1}{2V_p}\left[\int_\Omega \sqrt{\sigma^p_{ij}\sigma^p_{ij}}\, d\Omega\right]^2$$

subject to:

$$\operatorname{div}\left(\gamma_{ij} + \sum_{p=1}^{P}\sigma^p_{ij}\right) + \bar{\alpha}f = 0$$

$$\left(\gamma_{ij} + \sum_{p=1}^{P}\sigma^p_{ij}\right)\cdot n = \bar{\alpha}t \quad \text{on} \quad \Gamma_T,$$

$$\sigma^p \in K_p, p = 1,\ldots,P. \tag{P7}$$

This (convex) problem is a generalized minimum complementary energy statement that is applicable for a linear-softening material with a non-smooth energy functional, one that is not simply quadratic. However, the energy functional is homogeneous of degree two, meaning that the energy functional under proportional loading resembles the energy of an elastic-softening material with linear material components.

Problem (P7) is an equivalent problem statement, the solution of which is a distribution of stress fields σ^p, γ, which are stress fields of the *optimal* design. Note that the design parameters are hidden by the problem reduction. However, from the solution of (P7) the resource densities are given by (O1) and the material properties are recovered by (MAT).

This problem constitutes a generalization of the classical plastic design formulations for truss structures, the extension here covering linear-softening materials in a continuum setting. Its counterpart for truss structures with linear-softening material can be stated in the form:

$$\min_{\sigma^P,\gamma}\left\{(1/2V_0)\left[\sum_{i=1}^{m} a_i l_i|\gamma_i|\right]^2+\sum_{p=1}^{P}(1/2V_p)\left[\sum_{i=1}^{m} a_i l_i|\sigma_i^p|\right]^2\right\}$$

subject to:

$$B_{ij}\left(a_i\gamma_i+\sum_{p=1}^{P} a_i\sigma_i^p\right)=\bar{\alpha}f_j$$

$$|\sigma_i^p|\leqslant\bar{\sigma}^p, p=1,\ldots,P. \tag{P8}$$

This corresponds to a 'fully stressed' design formulation. In (P8), a_i and l_i denote the bar area and bar length, respectively, for the i th truss member, and the softening constraint is represented as a simple stress bound which is symmetric with respect to tension and compression. A derivation of (P8) through convex duality arguments is presented for the linearly-elastic case, expressed via a displacements based minimum compliance formulation, [Bendsøe *et al.* (1993)].

The computational results presented in this paper are obtained using a code for smooth optimization problems, to solve examples that are interpreted in the form of the (convex and smooth) problem (P6). The smoothness of (P6) as compared to (P7) is obtained at the expense of an increased number of variables. Alternatively, one can treat (P7) directly using non-smooth techniques, for example along the lines described by Allaire and Kohn (1993), Bendsøe *et al.* (1994) for similar types of problems. Instead of treating problem (P6) one can also revert to the form of the original problem of finding the maximal load carrying capacity, i.e. the problem:

$$\max_{\rho_p,\rho_0}\max_{\alpha,\sigma^P,\gamma}\alpha$$

subject to:

$$\operatorname{div}\left(\gamma_{ij}+\sum_{p=1}^{P}\sigma_{ij}^p\right)+\alpha f=0$$

$$\left(\gamma_{ij}+\sum_{p=1}^{P}\sigma_{ij}^p\right)\cdot n=\alpha t \quad \text{on} \quad \Gamma_T,$$

$$\sigma^p\in K_p, p=1,\ldots,P$$

$$\frac{1}{2}\int_\Omega\left(\frac{1}{\rho_0}\gamma_{ij}\gamma_{ij}+\sum_{p=1}^{P}\frac{1}{\rho_p}\sigma_{ij}^p\sigma_{ij}^p\right)\mathrm{d}\Omega\leqslant\Phi$$

$$\int_\Omega\Psi(F)\,\mathrm{d}\Omega\leqslant V_0;\int_\Omega\Psi(E^p)\,\mathrm{d}\Omega\leqslant V_p, p=1,\ldots P$$

$$0<\rho_0^{\min}\leqslant\rho_0\leqslant\rho_0^{\max}<\infty$$

$$0<\rho_p^{\min}\leqslant\rho_p\leqslant\rho_p^{\max}<\infty, p=1,\ldots,P. \tag{P9}$$

Bounds are imposed here on the range of variation of the resource variables in order to facilitate the computational work.

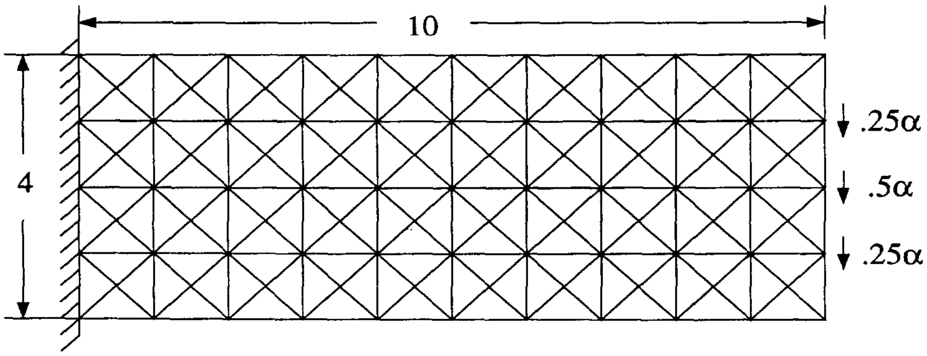


Fig. 1. Discretization and loads for the first example.

4. COMPUTATIONAL EXAMPLES OF DESIGN WITH SOFTENING MATERIAL

The computational results presented here demonstrate the design of material properties for several examples within plane stress modelling of the continuum. The optimization problem (P6) is transformed into a finite-dimensional nonlinear programming problem statement via a finite element discretization of stresses, deformations, and resource densities. The discretization model used is the simplest one possible, namely triangular elements with constant values for both element stresses and densities, and with equilibrium enforced in the weak sense at element corner nodes. A sequential quadratic programming subroutine [NLPQL, Schittkowski (1985)] is used to solve the inner minimization in (P6). This is coupled with an optimality criterion updating procedure applied to predict the resource density variables. An alternative and equally straightforward approach is to operate on the transformed version of (P6) directly, thus minimizing simultaneously on stresses and resource densities. Details regarding features of the solution procedures are given in Plaxton and Taylor (1993).

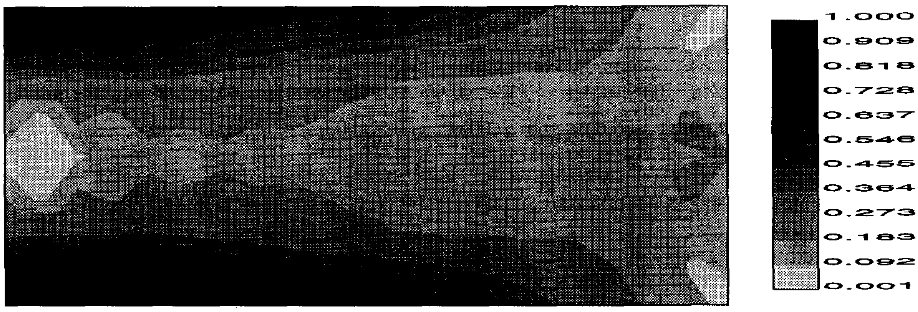
We note here that there is considerable possibility for improving the efficiency of the computational solution procedure. Given the convexity property of problem (P6), the problem could be solved using recently developed, efficient interior point methods. The FEM model is basic and unrefined, its purpose here being simply to serve for the production of example applications of our problem formulation. Results from these computations show the checkerboard type patterns usually associated with equal order approximations of the equilibrium and density fields, as discussed in Bendsøe *et al.* (1993) and in Jog *et al.* (1994). In the figures presented below, the pattern has been smeared out using a straightforward scheme for 'density averaging at nodal locations'. Finally, we note that the FEM discretized versions of problem (P6) will exhibit extensive sparsity, which could be exploited in order to improve computational efficiency.

The first example to be considered is the problem of an end-loaded cantilever system, discretized according to the schematic of Fig. 1. The material in this structure is represented by the linear stress component plus one softening constituent. The softening of the latter constituent is defined by the simple convex constraint

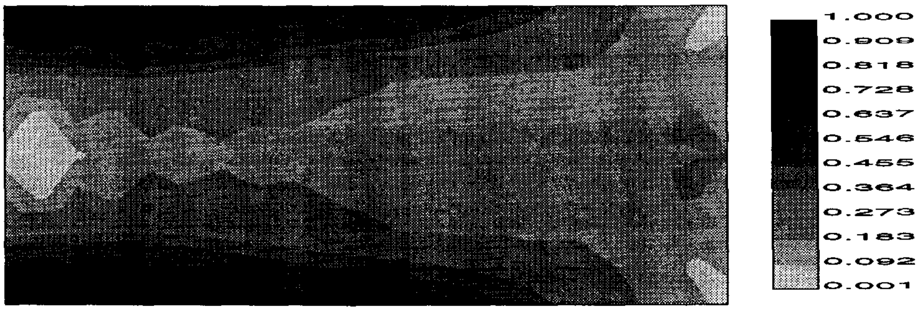
$$\sigma_{11}^2+\sigma_{22}^2+2\sigma_{12}^2-1.0 \leqslant 0.$$

The resource limit on each constituent is set at a value 0.3 times the resource value corresponding to a uniform distribution with $\rho = 1.0$. Lower and upper bounds on ρ, with values 0.001 and 1.0, are enforced as well. Results are presented for two values of the load parameter alpha, $\alpha = 0.1$ and $\alpha = 1.0$. At the lower value of load no softening occurs and so the system is entirely linear, while for $\alpha = 1.0$ the stress constraint is active over a substantial part of the structure.

Resource distributions for each constituent are presented for the lower load in Fig. 2 and in Fig. 3 for the greater load. Note that the distribution of the softening constituent is substantially more diffused for the higher loads. To provide for a visual interpretation of

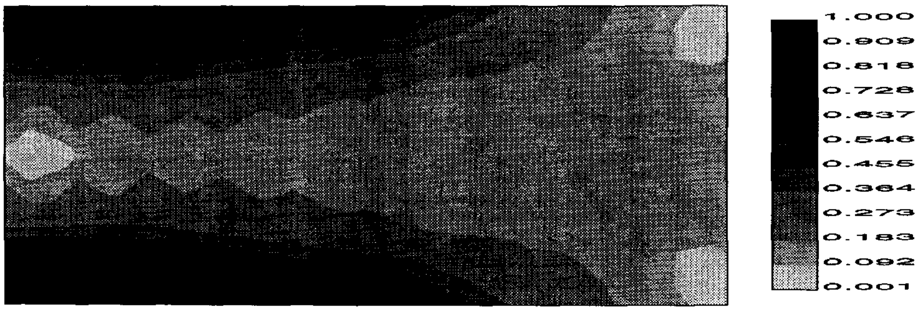


(a) Linear constituent resource distribution



(b) Limited constituent resource distribution

Fig. 2. Resource distribution contours for load $\alpha = 0.1$.

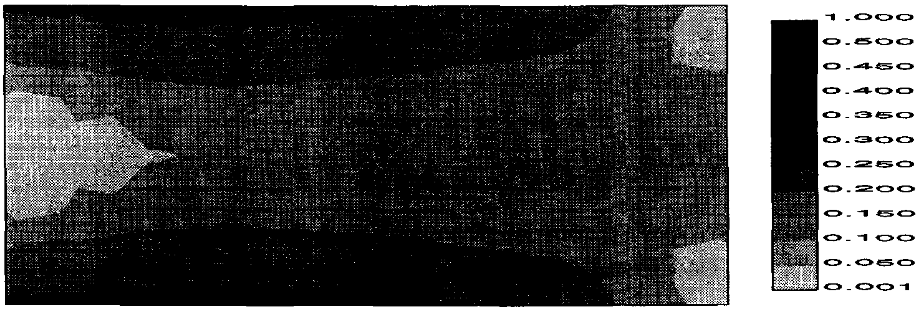


(a) Linear constituent resource distribution

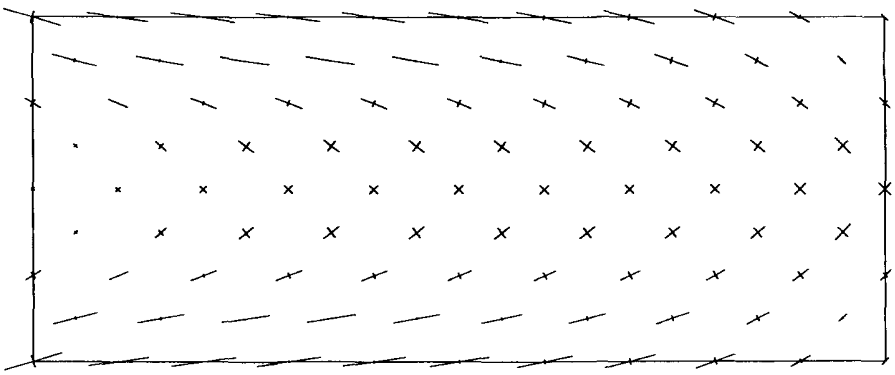
(b) Limited constituent resource distribution

Fig. 3. Resource distribution contours for load $\alpha = 1.0$.

(a) Resource distribution for the linear constituent



(b) Resource distribution for the softening constituent



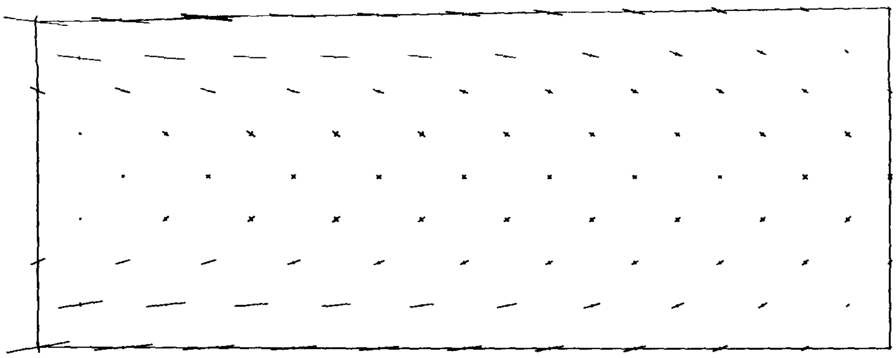
(c) Relative principal moduli and orientations for the softening constituent

Fig. 5. Results for load $\alpha = 1.0$ with reduced available resource.

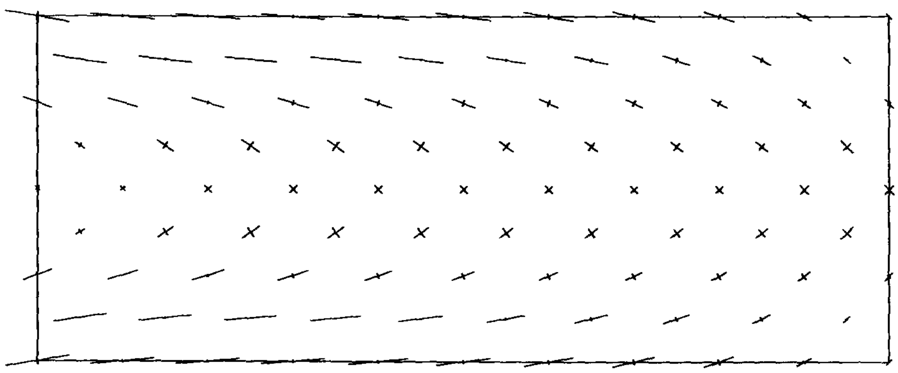
(a) Total resource distribution for load $\alpha = 0.1$

(b) Total resource distribution for load $\alpha = 10.0$

Fig. 7. Combined resource measures for design example with restricted domain.



(a) Load level $\alpha = 0.1$



(b) Load level $\alpha = 1.0$

Fig. 4. Relative moduli and corresponding directions for the optimal orthotropic softening constituent.

the (local) orthotropic structure of the optimal material properties, the magnitude and orientation of principal stresses in the softening constituent are shown (for both load levels) in Fig. 4. Principal stresses are oriented according to the crossed lines in these figures, and the lengths of the line segments measure relative magnitude of stress. Thus it can be recognized that both local and global distributions of resource in the softening constituent are depicted in Fig. 4. We simply note that the distribution of resource for the linear component is similar, for both loads, to what is shown in Fig. 4a.

For the purpose of comparison, the above example with load value $\alpha = 1.0$ is repeated, but with the resource constraint reduced to 0.1 times the value for which $\rho = 1.0$ over the entire structural domain. The resource distributions and the display of stresses for the softening constituent, which are shown in Fig. 5, indicate a slightly more sharply defined distribution for the linear constituent and a relatively more diffuse distribution for the softening one. Note that the contour levels in Fig. 5 differ from those in the first example, in order that the distribution of resource may be better visualized.

The final demonstration of the design of optimal material properties corresponds to a symmetrically loaded sheet with a central hole as shown in Fig. 6. The discretization for the quarter section analyzed is also shown in the figure. The resource limit factor on each constituent is again taken to be 0.3, and all other problem parameters are as in the first example. Load multiplier levels of 0.1 and 10.0 are considered. In this case, we display the total (combined ρ_0, ρ_1) density distribution for the two load levels, and a representative principal stress plot (for the total stress state at the higher load) in Figs 7 and 8.

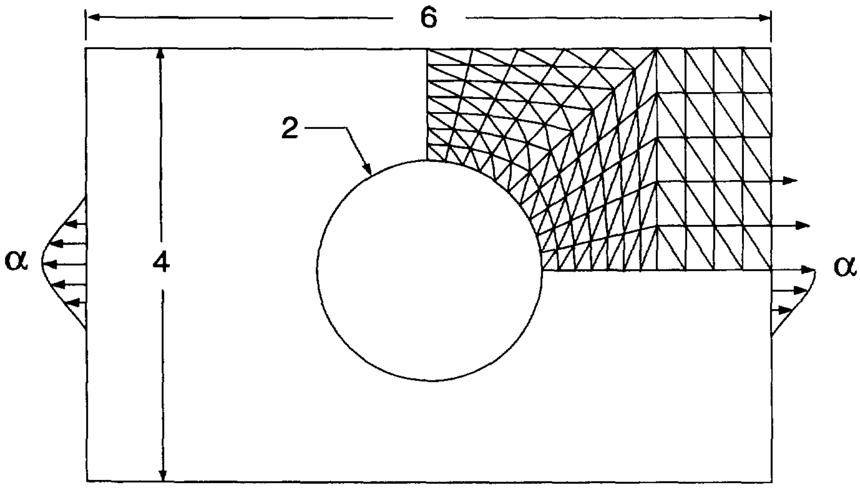


Fig. 6. Layout and discretization for design example with domain restriction.

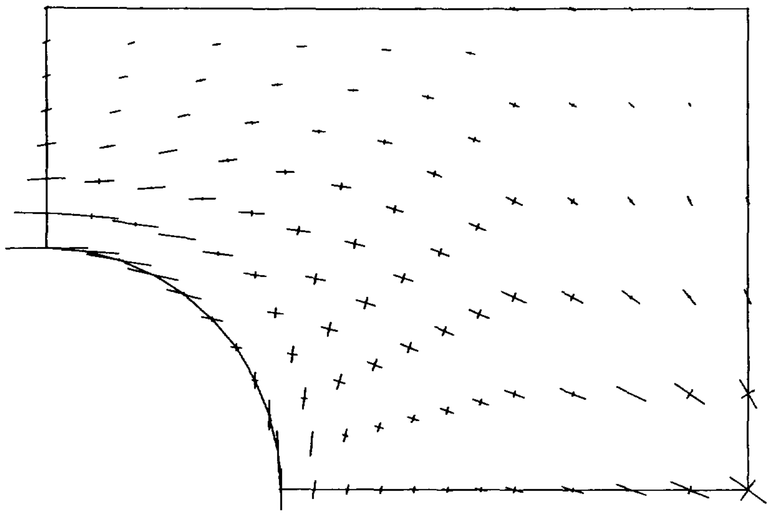


Fig. 8. Principal directions and relative magnitudes of total stress at load level $\alpha = 10$.

5. CONCLUSIONS

We have demonstrated that the design of material can be extended to a general class of analysis situations encompassing structures made of elastic/softening materials. The optimal material properties can be derived analytically, and this provides for a considerable simplification in the analysis and a commensurate reduction in problem size. The analysis applies as well in two and three-dimensions, with the reduction in problem size being especially important in the three-dimensional setting, particularly to render the computational problem into tractable size.

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APPENDIX

We wish to solve the problem (P5)

$$\inf_{E>0,\Psi(E)=\rho} E^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}$$

for any stress field σ_{ij}.

The constraints imply that all the positive eigenvalues of any admissible tensor E are less than ρ. This in turn means that we have the lower bound

$$\inf_{E>0,\Psi(E)=\rho} E^{-1}_{ijkl}\sigma_{ij}\sigma_{kl} \geqslant \frac{1}{\rho}\sigma_{ij}\sigma_{ij}. \tag{A1}$$

To prove that we in fact have equality in (A1) we will construct a sequence E^{ε} of admissible tensors for which

$$\lim_{\varepsilon\to 0}[E^{\varepsilon^{-1}}]_{ijkl}\sigma_{ij}\sigma_{kl} = \frac{1}{\rho}\sigma_{ij}\sigma_{ij}. \tag{A2}$$

For this, let $\gamma^1,\ldots,\gamma^N$ ($N = 3$ in dimension 2 and $N = 6$ in dimension 3) be an orthonormal basis of symmetric 2-tensors, with $\gamma^1_{ij} = \sigma_{ij}/\sqrt{\sigma_{pq}\sigma_{pq}}$. A sequence E^{ε} of admissible, positive definite tensors satisfying (A2) can then be constructed by setting

$$E^{\varepsilon}_{ijkl} = (\rho-\varepsilon)\gamma^1_{ij}\gamma^1_{kl}+\delta\gamma^2_{ij}\gamma^2_{kl}+\cdots+\delta\gamma^N_{ij}\gamma^N_{kl} \tag{A3}$$

where δ is chosen so $\Psi(E^{\varepsilon}) = \rho$ and where $0 < \varepsilon < \rho$. For E^{ε} we have that

$$[E^{\varepsilon^{-1}}]_{ijkl}\sigma_{ij}\sigma_{kl} = \left[\frac{1}{(\rho-\varepsilon)}\gamma^1_{ij}\gamma^1_{kl}+\frac{1}{\delta}\gamma^2_{ij}\gamma^2_{kl}+\cdots+\frac{1}{\delta}\gamma^N_{ij}\gamma^N_{kl}\right]\sigma_{ij}\sigma_{kl} = \frac{1}{\rho-\varepsilon}\sigma_{ij}\sigma_{ij}$$

showing that (A2) holds and thus proving formula (P5).